Lecture 2. **Dynamic equations. Mathematical description of systems in the state space. Linearization**

2.1. The linear systems of differential equations

A general control object is a dynamic system the properties of which can be described by ordinary differential equations. In this connection control theory faces the *identification problem***:** how to obtain a mathematical description of a dynamic system? Nowadays we can find sufficiently accurate solutions of a parametric identification problem having defined a mathematical description structure, i.e. a mathematical model.

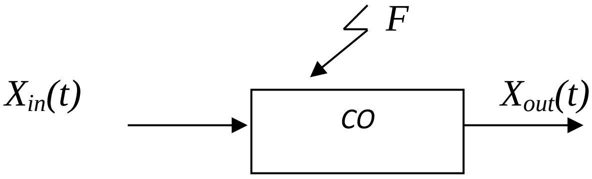

Fig. 2.0. Control Object with undefined point of application of external disturbance

Let a system behavior be completely described by an ordinary differential equation of an order "*n*" with constant coefficients:

$$a_0 \frac{d^n X_{out}}{dt^n} + a_1 \frac{d^{n-1} X_{out}}{dt^{n-1}} + \ldots + a_n X_{out}(t) = b_0 \frac{d^m X_{in}}{dt^m} + b_1 \frac{d^{m-1} X_{in}}{dt^{m-1}} + \ldots + b_m X_{in}(t) \ . \qquad (2.1)$$

Here $a_i \ (i = \overline{0,n}),\ b_j \ (j = \overline{0,m})$ are constants; $m \le n$ defines physical realizability condition.

To obtain a solution of an ordinary differential equation of order "*n*" we need exactly "*n*" initial conditions predefined:

$$X_{out}(0) \neq 0;\ \dot{X}_{out}(0) \neq 0;\ \ddot{X}_{out}(0) \neq 0;\ \ldots ;\ X^{(n-1)}{}_{out}(0) \neq 0 \, .$$

A complete solution of an ordinary differential equation is composed as a sum of general and particular *solutions:*

$$X_{out}(t) = X_{out}^{gen}(t) + X_{out}^{part}(t) \, .$$

A general solution of an ordinary differential equation is a sum of exponents:

$$X_{out}^{gen}(t) = \sum_{i=1}^{n} c_i e^{\lambda_i t} \ . \qquad (2.2)$$

Here (in 2.2) c_i *(i = 1,..., n)* are the constants defined by the initial conditions. Here λ_i $(i = \overline{1, n})$ are proper numbers calculated as a solution of characteristic polynomial of the following form:

$$a_0\lambda^n + a_1\lambda^{n-1} + \ldots + a_{n-1}\lambda + a_n = 0.$$

A general solution (2.2) characterizes *proper movement (free movement)* of a dynamic system.

A Particular solution has the form

$$X_{out}^{part}(t) = \int_0^t K(t-\tau)X_{in}(\tau)d\tau \ , \tag{2.3}$$

where function *K(t - τ)* is a weight function. The *particular solution* characterizes *forced movement* of a system.

Let us compare mathematical and control theory points of view on solution of an ordinary differential equation (fig. 2.1).

Mathematic	⇒	*general*	+	*particular*
		⇓		⇓
Control System	⇒	*proper movement (transient process)*	+	*forced movement (steady-state)*

Fig. 2.1. Two different points of view on one solution of an ordinary differential equation

As you can see in control theory free movement is called *transient process*, and forced movement is called *steady-state*. The later formally is a solution of a differential equation after initial conditions took place.

If the right-hand side of education (2.1) changes badly (i.e. it involves derivatives of low order) the left-hand side changes badly too. It is not the best way one would try to obtain really good mathematical description since the input is reflected almost unchanged into the output. To provide rapidly varying signal the input should be a subject to significant changes, step function is a good example in this case. Yet as a consequence the derivative has point of discontinuity so mathematical description is taken under consideration only in narrow range of values, providing only partial solutions. This fact makes us always specify the range of values in which mathematical description is applicable.

Next, having the description (2.1) we also need to obtain solution that will show us a trajectory of system movement. For this purpose we need to integrate (2.1) "*n*" times, or move from differential equation of an order "*n*" to "*n*" first order differential equations (Cauchy form is the right choice) and solve the resulting system.

Consider homogeneous system of the first order differential equations in the following form:

$$\begin{cases} \dot{x}_1 = a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\ \dot{x}_2 = a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \dot{x}_n = a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n \end{cases} . \qquad (2.4)$$

The system (2.4) can be rewritten in matrix form as:

$$\dot{X} = AX . \qquad (2.4a)$$

Here the matrix A is equal $A(n\times n) = \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{vmatrix}$; the vectors $X(n\times 1) = \begin{vmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{vmatrix}$, $\dot{X}(n\times 1) = \begin{vmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \ldots \\ \dot{x}_n \end{vmatrix}$.

Now we will write solution of the system (2.4). It can be written as (2.2), since it is the direct representation of (2.1), or in matrix form:

$$X(t) = e^{At}X_0 \quad , \qquad (2.5)$$

where vector X_0 is the value of vector X at initial moment of time $t=t_0=0$, $X(0)\neq 0$.

In (2.5) exponential matrix e^{At} is expanded into an infinite consistent as the following:

$$e^{At} = I + At + \frac{A^2}{2!}t^2 + \frac{A^3}{3!}t^3 + \ldots + \frac{A^n}{n!}t^n + \ldots \qquad (2.6)$$

and is called a *transfer matrix* of the system (2.4) or (2.4a) in the matrix form.

Consider nonhomogeneous system of the first order differential equations, i.e. the case when nonzero input signal $U(t)$ is applied to (2.4):

$$\begin{cases} \dot{x}_1 = a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n + b_1U \\ \dot{x}_2 = a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n + b_2U \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \dot{x}_n = a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n + b_nU \end{cases} \tag{2.7}$$

Introduce a new vector $b^T = (b_1, b_2, \ldots, b_n)$ and rewrite (2.7) in the matrix form as

$$\dot{X} = AX + b^T U \ . \tag{2.7a}$$

Let at initial moment of time $t=t_0=0$ the value vector $X(t)$ be equal to $X_0 \neq 0$, then the solution of the system (2.7) in the matrix form will be written as

$$X(t) = e^{At}X_0 + \int_{t_0}^{t_1} e^{A(t-\tau)} bU(\tau)d\tau \, .$$

In this equation $e^{At}X_0$ is *a general solution* while $\int_{t_0}^{t_1} e^{A(t-\tau)} b^T U(\tau)d\tau$ is *a particular solution.*

2.2 Mathematical description of ACS in state space

A state space mathematical description (or a mathematical model) is presented in general as the following:

$$\begin{cases} \dot{X}_1 = f_1(x_1, x_2, \ldots, x_n, U_1, U_2, \ldots, U_m) \\ \dot{X}_2 = f_2(x_1, x_2, \ldots, x_n, U_1, U_2, \ldots, U_m) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \dot{X}_n = f_n(x_1, x_2, \ldots, x_n, U_1, U_2, \ldots, U_m) \end{cases} , \tag{2.8}$$

where X_i are state variables *(i = 1,…, n)* and U_j are control variables *(j = 1,…, m).* The equations in system (2.8) are called *state equations* or constitutive equations.

Denote all system output variables as y_k *(k = 1, … , r),* then:

$$\begin{cases} y_1 = \varphi_1(x_1, x_2, \ldots, x_n) \\ y_2 = \varphi_2(x_1, x_2, \ldots, x_n) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ y_r = \varphi_r(x_1, x_2, \ldots, x_n) \end{cases} . \tag{2.9}$$

Now a control object in state space is depicted as in fig. 2.2.

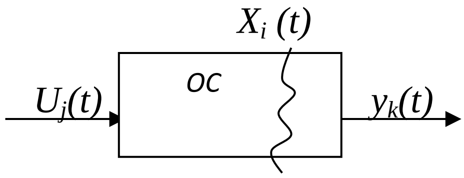

Fig. 2.2. Control object in state space

The functions f_i *(i = 1,…, n)* and φ_k *(k = 1,…, r)* reflect mathematical presentation of physical laws obeyed by a particular control object.

If the control U_j impacts directly on outputs y_k then output variables equations have the following form:

$$\begin{cases} y_1 = \varphi_1(x_1, x_2, ..., x_n, U_1, U_2, ..., U_m) \\ y_2 = \varphi_2(x_1, x_2, ..., x_n, U_1, U_2, ..., U_m) \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ y_r = \varphi_r(x_1, x_2, ..., x_n, U_1, U_2, ..., U_m) \end{cases}. \tag{2.9a}$$

The equations in systems (2.9) and (2.9a) are often called *observer able equations*. Let us introduce new vector entities, i.e. state a variable vector, a control variable vector and an output variable vector:

$$X(n\times 1) = \begin{vmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{vmatrix}, \qquad U(m\times 1) = \begin{vmatrix} u_1 \\ u_2 \\ \cdots \\ u_m \end{vmatrix}, \qquad y(r\times 1) = \begin{vmatrix} y_1 \\ y_2 \\ \cdots \\ y_r \end{vmatrix}.$$

Having done all this we at last can rewrite our system equations in the purely vector form:

$$\begin{cases} \dot{X} = f(X, U) \\ Y = \varphi(X) \end{cases} \quad \text{or} \quad \begin{cases} \dot{X} = f(X, U) \\ Y = \varphi(X, U) \end{cases}. \tag{2.10}$$

Here functions *f(·)* and *φ(·)* are nonlinear vector functions of vector arguments *X* and *U*: $f(n\times 1) = \begin{vmatrix} f_1 \\ f_2 \\ \cdots \\ f_n \end{vmatrix}$ and $\varphi(r\times 1) = \begin{vmatrix} \varphi_1 \\ \varphi_2 \\ \cdots \\ \varphi_r \end{vmatrix}$.

This completes state space description of ACS. But there is one disturbing fact: it is not uncommon to have differential equations mentioned so far in nonlinear form.

They are hard to analyze and even harder to solve synthesis. Therefore in such cases we first of all have to linearize the given differential equations to obtain the first approximation equations that are much simpler to analyze and solve synthesis.

2.3 Linearization

Let a dynamic system be described by equations $\dot{X} = f(X,U)$ and $Y = \varphi(X,U)$. Consider equations in the form (2.8) and (2.9a) and expand nonlinear functions x_i, y_k $(i=\overline{1,n}, \quad k=\overline{1,r})$ into Taylor series in the vicinity of operation condition:

$$\begin{aligned} &X_i^{\,0} \neq 0 \qquad (i=\overline{1,n}); \\ &U_j^{\,0} \neq 0 \qquad (j=\overline{1,m}). \end{aligned} \tag{*}$$

We will place all terms of the order second and higher group into R_i, F_k $(i=\overline{1,n}, \quad k=\overline{1,r})$.

Then the state equations system will take the following form:

$$\begin{cases} \dot{X}_1 = a_{11}X_1 + a_{12}X_2 + \ldots + a_{1n}X_n + b_{11}U_1 + b_{12}U_2 + \ldots + b_{1m}U_m + R_1 \\ \dot{X}_2 = a_{21}X_1 + a_{22}X_2 + \ldots + a_{2n}X_n + b_{21}U_1 + b_{22}U_2 + \ldots + b_{2m}U_m + R_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \dot{X}_n = a_{n1}X_1 + a_{n2}X_2 + \ldots + a_{nn}X_n + b_{n1}U_1 + b_{n2}U_2 + \ldots + b_{nm}U_m + R_n \end{cases} . \tag{2.11}$$

Accordingly, the observer able equations system will be presented as:

$$\begin{cases} y_1 = c_{11}X_1 + c_{12}X_2 + \ldots + c_{1n}X_n + d_{11}U_1 + d_{12}U_2 + \ldots + d_{1m}U_m + F_1 \\ y_2 = c_{21}X_1 + c_{22}X_2 + \ldots + c_{2n}X_n + d_{21}U_1 + d_{22}U_2 + \ldots + d_{2m}U_m + F_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ y_r = c_{r1}X_1 + c_{r2}X_2 + \ldots + c_{rn}X_n + d_{r1}U_1 + d_{r2}U_2 + \ldots + d_{rm}U_m + F_r \end{cases} . \tag{2.12}$$

Here $a_{ij} = \left(\dfrac{\partial X_i}{\partial X_j}\right)^0$, $b_{ij} = \left(\dfrac{\partial X_i}{\partial U_j}\right)^0$, $c_{kj} = \left(\dfrac{\partial Y_k}{\partial X_j}\right)^0$, $d_{kj} = \left(\dfrac{\partial Y_k}{\partial U_j}\right)^0$, $i=\overline{1,n}$; $j=\overline{1,m}$; $k=\overline{1,r}$; they are constants; their values are defined as values of corresponding partial derivatives computed at the point (*).

If differences of variables X_i, U_j between actual values of X_i and U_j and their values at (*) $\left(X_i - X_i^{\,0}\right)$, $\left(U_j - U_j^{\,0}\right)$ are sufficiently small, then in (2.11) and (2.12) we can neglect nonlinear terms Ri and Fk since their higher order of vanishing. Then we can rewrite the systems:

$$\begin{cases} \dot{X}_1 = a_{11}X_1 + a_{12}X_2 + \ldots + a_{1n}X_n + b_{11}U_1 + b_{12}U_2 + \ldots + b_{1m}U_m \\ \dot{X}_2 = a_{21}X_1 + a_{22}X_2 + \ldots + a_{2n}X_n + b_{21}U_1 + b_{22}U_2 + \ldots + b_{2m}U_m \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \dot{X}_n = a_{n1}X_1 + a_{n2}X_2 + \ldots + a_{nn}X_n + b_{n1}U_1 + b_{n2}U_2 + \ldots + b_{nm}U_m \end{cases} \quad (2.13)$$

$$\begin{cases} y_1 = c_{11}X_1 + c_{12}X_2 + \ldots + c_{1n}X_n + d_{11}U_1 + d_{12}U_2 + \ldots + d_{1m}U_m \\ y_2 = c_{21}X_1 + c_{22}X_2 + \ldots + c_{2n}X_n + d_{21}U_1 + d_{22}U_2 + \ldots + d_{2m}U_m \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ y_r = c_{r1}X_1 + c_{r2}X_2 + \ldots + c_{rn}X_n + d_{r1}U_1 + d_{r2}U_2 + \ldots + d_{rm}U_m \end{cases} . \quad (2.14)$$

The equations in systems (2.13) and (2.14) are *linearized-state* and *observer able equations correspondingly;* they are called *linear approximation equations*.

Introduce several new matrixes:

$$A = \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{vmatrix}, \quad B = \begin{vmatrix} b_{11} & b_{12} & \ldots & b_{1m} \\ b_{21} & b_{22} & \ldots & b_{2m} \\ \ldots & \ldots & \ldots & \ldots \\ b_{n1} & b_{n2} & \ldots & b_{nm} \end{vmatrix},$$

$$C = \begin{vmatrix} c_{11} & c_{12} & \ldots & c_{1n} \\ c_{21} & c_{22} & \ldots & c_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ c_{r1} & c_{r2} & \ldots & c_{rn} \end{vmatrix}, \quad D = \begin{vmatrix} d_{11} & d_{12} & \ldots & d_{1m} \\ d_{21} & d_{22} & \ldots & d_{2m} \\ \ldots & \ldots & \ldots & \ldots \\ d_{r1} & d_{r2} & \ldots & d_{rm} \end{vmatrix}.$$

Then (2.13) and (2.14) can be rewritten in a matrix form as:

$$\begin{cases} \dot{X} = AX + BU \\ Y = CX + DU \end{cases} . \quad (2.15)$$

If a particular ACS does not have direct impact of control signal upon output, the system (2.15) becomes slightly simpler:

$$\begin{cases} \dot{X} = AX + BU \\ Y = CX \end{cases} . \quad (2.16)$$

In the nearest future we will use this particular dynamic system description